

Amplification of External EM-wave by Nonlinear Wake Waves in Cold Plasma.

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1 Abstract

Interaction of external monochromatic, linearly polarized plane EM-wave with nonlinear one-dimensional wake wave, generated by relativistic electron bunch moving in cold plasma, is considered.

At definite conditions on parameters of plasma and bunch, nonlinear wake wave can have a pronounced spikes, where plasma electron density has its maximum value and plasma electron velocities are nearing wave breaking limit, i.e. velocity of the rigid driving bunch [1]-[5]. Presented calculations show, that external EM-wave, propagating through plasma, normal to the bunch velocity, at such a conditions can be amplified inside the spikes, due to interaction with the plasma electrons. EM-wave, Thomson scattered on spikes, is also amplified. Amplification factors are obtained for the both cases at different conditions on EM-wave-plasma-electron bunch system parameters. An additional amplification takes place, when the frequency of EM-wave is nearing to the spike plasma frequency.

Results are obtained in the perturbative approach, when dimensionless external EM-wave amplitude is used as a small parameter. The exact one-dimensional solution of the problem of nonlinear wake wave generation in cold plasma by relativistic electron bunch, obtained previously in [4], [5], is taken as a zero order approximation.

2 Introduction

The problem of interaction of external electromagnetic wave (EM-wave) with electrons or electron bunches, moving in plasma, and with plasma wake

waves, generated by these electron beams, have been a subject of numerous investigations [6]-[10]. It was assumed that driving electrons or electron bunches are nonrelativistic, generated plasma wake waves are linear and physical problems studied in [6]-[10] have been devoted to electron ionization losses and electron acceleration.

Plasma - relativistic electron bunch RF - generators have been proposed in [11] and experimentally investigated in number of works (see reviews [12] - [14]). Nonlinear Thomson scattering of intense laser pulses from plasmas was considered in [15].

In the present work considered plasma wake waves are essentially nonlinear, generated by relativistic rigid electron bunches and external EM-wave is weak enough in order to provide the possibility to use the perturbative approach. The goal of present investigation is to find out the possibility and conditions, when external EM-wave can be amplified inside the plasma as well as Thomson scattered EM-wave. Main difference from previous investigations consists in consideration the generated plasma waves, which are near the wave breaking limit. It is shown that if some conditions near to wake wave breaking limit are satisfied, the essential amplifications in both considered cases can take place.

The breaking of waves is one of the most impressive phenomena, where effect of nonlinearities stands out of clearly. Plasma wave breaking was predicted in the works of Akhieser and Polovin [1] and Dawson [2] (see also [3]). The connection of the parameters of plasma wake wave breaking limit with the parameters of plasma and driving relativistic electron bunch in one-dimensional approach have been obtained in [4], [5].

Recently, the variety of nonlinear plasma wave-breaking was considered in the frame of laser wake field acceleration (LWFA) concept (see e.g. [16]-[18]). Wake wave breaking has followed the so called "blow out" regime, [19] introduced in the frame of plasma wake field acceleration (PWFA) concept (see also review [18]). As it will be seen from the presented results, wake waves near wavebreaking limit can also be used for EM-wave amplification with possible application to design the powerful klystron type EM-wave amplifiers or generators for future linear colliders with high acceleration gradient.

3 Formulation of the Problem and Basic Equations

Consider flat electron bunch, moving in cold plasma, along z-direction with the velocity v_0 in lab system. Longitudinal length of the bunch is d , transverse dimensions of the bunch are taken infinite, i.e. the problem can be treated, in the absence of external electromagnetic (EM) field, as one dimensional. Bunch is placed at the moment of time $t = 0$ in interval $0 \leq z \leq d$, ions are considered as immobile, plasma electron density at equilibrium is n_0 , plasma linear frequency $\omega_p^2 = 4\pi e^2 n_0 / m$.

One dimensional approaches valid for wide enough bunches, when bunch radius $r_0 \gg \frac{c}{\omega_p}$ [20]. The condition could be relaxed by at same sence more adjustable one, if external constant longitudinal strong enough magnetic field H_0 is applied to the system. Then one dimensionality condition would be $r_0 \gg r_L, \Omega_l \gg \omega_p$, where $r_L = \frac{v_{tr}}{\Omega_L} = \frac{cp_{tr}}{cH_0}, \Omega_L = \frac{ecH_0}{\epsilon}$ are Larmour radius and frequency subsequently for plasma electrons with energy \mathcal{E} , transverse velocity and momenta v_{tr}, p_{tr} .

The considered cold plasma - rigid electron bunch system is interacting with the external monochromatic, linearly polarized electromagnetic wave (EM-wave), propagating through plasma in x direction, perpendicular to bunch velocity and with electric vector directed along 0z - axis (p - polarization):

$$\begin{aligned} \mathcal{E}_z &= \mathcal{E}_{0z} e^{-i\omega_0 t + ik_0 x} \\ \mathcal{H}_y &= -\sqrt{\epsilon} \mathcal{E}_{z0} e^{-i\omega_0 t + ik_0 x}, k_0 = \sqrt{\epsilon} \frac{\omega_0}{c}, \end{aligned} \quad (1)$$

where ϵ is a plasma dielectric constant

$$\epsilon = \epsilon' + i\epsilon'' = 1 - \frac{\varpi_p^2}{\omega_0^2} + i \frac{\varpi_p^2 \nu_{eff}}{\omega_0^3} \quad (2)$$

and $\varpi_p^2 = \frac{4\pi e^2 n^{(0)}(z)}{m}$, $n^{(0)}(z)$ is the plasma electron density in wake wave, generated by driving bunch in the absence of external EM-wave, ν_{eff} - effective collisions frequency of plasma electrons, $\nu_{eff} \ll \varpi_p < \omega_0$.

The considered problem consists in finding out the components of EM - field with frequency ω_0 inside and outside (Thomson scattered) of the plasma, which is modification of the external field (1) due to interaction with the

plasma electrons, perturbed by relativistic rigid electron bunch, moving in cold plasma. The back influence of plasma wake wave on the bunch [21] as well as bunch interaction with external EM - wave are disregarded, due to assumption than driving bunch is relativistic enough.

It is assumed also, that considered plasma column has a thickness Δ along the direction of EM - wave propagation (0x - axis), which is smaller than corresponding skin length of plasma $\Delta < \frac{\sqrt{2}c}{\omega_0} \sqrt{|\epsilon''|}$.

Introduce the dimensionless arguments:

$$t' = \omega_p t, x'_1 z' = k_p x, k_p z, k_p = \omega_p / c, \quad (3)$$

and dimensionless variables:

$$\begin{aligned} \vec{E} &= \frac{m\omega_p c}{e} \vec{E}', \vec{H} = \frac{m\omega_p c}{e} \vec{H}', \\ \frac{n_e}{n_0} &= n', \frac{n_b}{n_0} = n'_b, \\ \beta_0 &= \frac{v_0}{c}, \vec{\beta} = \frac{\vec{v}_e}{c}, \vec{\rho} = \frac{\vec{p}_e}{mc} \\ \mathcal{E}'_z &= \frac{m\omega_p c}{e} \mathcal{E}_z, \mathcal{H}'_y = \frac{m\omega_p c}{e} \mathcal{H}_y, \\ \mathcal{E}'_z &= a \frac{\omega_0}{\omega_p} e^{-i\frac{\omega_0}{\omega_p} t' + i\frac{k_0}{k_p} x'}, \\ \mathcal{H}'_y &= -a \frac{\omega_0}{\omega_p} \sqrt{\epsilon} e^{-i\frac{\omega_0}{\omega_p} t' + i\frac{k_0}{k_p} x'}, \\ a &\equiv \frac{e\mathcal{E}_{0z}}{mc\omega_0} = \frac{e\mathcal{E}_{z0}}{mc\omega_p} \frac{\omega_p}{\omega_0}, \end{aligned} \quad (4)$$

where \vec{E}, \vec{H} are EM - field generated in plasma by bunch and plasma electrons, $\vec{v}_e, \vec{p}_e, n_e$ - velocity, momenta and density of plasma electrons, a - dimensionless amplitude of the external EM - wave.

The considered system of cold plasma - rigid relativistic electron bunch - external EM - wave is described by the following set of equations (where primes are temporally omitted):

$$\begin{aligned} 1. \quad \frac{\partial \vec{\rho}}{\partial t} + \left(\vec{\beta} \frac{\partial}{\partial \vec{r}} \right) \vec{\rho} &= - \left(\vec{\mathcal{E}} + [\vec{\beta} \mathcal{H}] \right) - \left(\vec{E} + [\beta \vec{H}] \right) \\ 2. \quad \text{div} \vec{E} &= 1 - n - n_b \end{aligned} \quad (5)$$

$$\begin{aligned}
3. \quad & \frac{\partial n}{\partial t} + \text{div} (n\vec{\beta}) = 0 \\
4. \quad & \text{rot}\vec{H} = \frac{\partial \vec{E}}{\partial t} - n\vec{\beta} - n_b\vec{\beta}_0 \\
5. \quad & \text{rot}E = -\frac{\partial \vec{H}}{\partial t}
\end{aligned}$$

Continuity eq. (5.3) is follows from (5.2) and (5.4). For convinience in what follows all equations (5) are used.

System (5) will be solved by perturbative approach, when dimensionless external EM - wave amplitude a is taken as a small parameter, $a \ll 1$. In the zero approximation ($a = 0, \mathcal{E} = 0, \mathcal{H} = 0$) the system of equation (5) reduces to the set of the following equations, describing the cold plasma - one dimensional relativistic electron bunch system, which were exactly solved in [4],[5], assuming the realization of steady state regime (which means that all variables are function of $\tilde{z} = z - \beta_0 t$ only):

$$\begin{aligned}
1. \quad & (\beta_0 - \beta_z^{(0)}) \frac{\partial \rho_z^{(0)}}{\partial \tilde{z}} = E_z^{(0)} \\
2. \quad & \frac{\partial E_z^{(0)}}{\partial \tilde{z}} = 1 - n^{(0)}(\tilde{z}) - n_b \\
3. \quad & \frac{d}{d\tilde{z}} [n^{(0)} (\beta_0 - \beta_z^{(0)})] = 0
\end{aligned} \tag{6}$$

($\vec{H} = 0$ due to the symmetry of the problem).

In (6) superscript (0) denote the subsequent variables at zero order approximation ($a = 0$).

From eq. (6.3) with the boundary condition $n^{(0)} = n_0, \beta_z^{(0)} = 0$, when $\tilde{z} = d$ (on the front of the moving bunch) it follows that

$$\begin{aligned}
n^{(0)}(\tilde{z}) &= \frac{\beta_0}{\beta_0 - \beta_z^{(0)}(\tilde{z})} = \frac{\beta_0(1 + \rho_z^{(0)2})^{1/2}}{\beta_0(1 + \rho_z^{(0)2})^{1/2} - \rho_z^{(0)}} > 0, \\
\beta_z^{(0)} &= \frac{\rho_z^{(0)}}{(1 + \rho_z^{(0)2})^{1/2}} < \beta_0,
\end{aligned} \tag{7}$$

(this result also can be obtained from eqs. (5.2), (5.4) written in zero approximation).

Inside the driving bunch $\beta_z^{(0)}$ is always negative, so $n^{(0)} > 0$ automatically; behind the bunch, in wake waves, $\beta_z^{(0)}$ periodically changes sign and at some \tilde{z} , when $\beta_z^{(0)}$ is $0 < \beta_z^{(0)} \rightarrow \beta_{max} < \beta_0, \tilde{z} \rightarrow \tilde{z}_m$ the plasma electron density $n^{(0)}(z_m)$ could be large enough, nearing wave breaking limit. The values

of β_{max} (or ρ_{max}) could be expressed through plasma and driving bunch parameters $n_0, n_b, d, \gamma_0 = (1 - \beta_0^2)^{-1/2}$ using results of the work [5].

For consideration of the next approximations to the set (5) introduce a new arguments

$$\tilde{z} = z - \beta_0 t, \tau = t, x = x.$$

Then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \beta_0 \frac{\partial}{\partial \tilde{z}}, \frac{\partial}{\partial z} = \frac{\partial}{\partial \tilde{z}}, \quad (8)$$

Variables in eqs. (5) decompose in the following series

$$\begin{aligned} \rho_z &= \rho_z^{(0)}(\tilde{z}) + \epsilon \rho_z^{(1)}(\tilde{z}, x, \tau) + \dots, \\ \rho_x &= \epsilon \rho_x^{(1)}(\tilde{z}, x, \tau) + \dots, \\ \beta_z &= \beta_z^{(0)}(\tilde{z}) + \epsilon \beta_z^{(1)}(\tilde{z}, x, \tau) + \dots, \\ \beta_x &= \epsilon \beta_x^{(1)}(\tilde{z}, x, \tau) + \dots, \\ E_z &= E_z^{(0)}(\tilde{z}) + \epsilon E_z^{(1)}(\tilde{z}, x, \tau) + \dots, \\ E_x &= \epsilon E_x^{(1)}(\tilde{z}, x, \tau) + \dots, \\ H_y &= \epsilon H_y^{(1)}(\tilde{z}, x, \tau) + \dots, \end{aligned} \quad (9)$$

where $\epsilon \equiv a$; in what follows ϵ will be put equal 1 and a will be included in variables $\rho^{(1)}, E^{(1)}, \dots$

The variables $\vec{\beta}^{(1)}$ and $\vec{\rho}^{(1)}$ are connected by the following approximation relation valid, when $|\rho_z^{(1)}| \ll |\rho_z^{(0)}|, |\rho_x^{(1)}| \ll |\rho_x^{(0)}|$, up to $O(\epsilon)$

$$\beta_z^{(1)} = \frac{\rho_z^{(1)}}{1 + \rho_z^{(0)2}}, \beta_x^{(1)} = \frac{\rho_x^{(1)}}{(1 + \rho_z^{(0)2})^{1/2}} \quad (10)$$

In the first approximation ($\sim a$) the following set of equations is obtained from (5), (8), (9):

$$\begin{aligned} 1.1. \quad & \frac{\partial \rho_z^{(1)}}{\partial \tau} - (\beta_0 - \beta_z^{(0)}) \frac{\partial \rho_z^{(1)}}{\partial \tilde{z}} + \beta_z^{(1)} \frac{\partial \rho_z^{(0)}}{\partial \tilde{z}} = -E_z^{(1)} - \mathcal{E}_z \\ 1.2. \quad & \frac{\partial \rho_x^{(1)}}{\partial \tau} - (\beta_0 - \beta_z^{(0)}) \frac{\partial \rho_x^{(1)}}{\partial \tilde{z}} = -E_x^{(1)} + \beta_z^{(0)} (\mathcal{H}_y + H_y^{(1)}) \\ 2. \quad & \frac{\partial E_x^{(1)}}{\partial x} + \frac{\partial E_z^{(1)}}{\partial \tilde{z}} = -n^{(1)} \\ 3. \quad & \frac{\partial n^{(1)}}{\partial \tau} - (\beta_0 - \beta_z^{(0)}) \frac{\partial n^{(1)}}{\partial \tilde{z}} + \frac{\partial}{\partial \tilde{z}} (n^{(0)} \beta_z^{(1)}) + \frac{\partial}{\partial x} (n^{(0)} \beta_x^{(1)}) = 0 \\ 4.1. \quad & \frac{\partial H_y^{(1)}}{\partial x} = -\frac{\partial E_z^{(1)}}{\partial \tau} - \beta_0 \frac{\partial E_z^{(1)}}{\partial \tilde{z}} - n^{(0)} \beta_z^{(1)} - n^{(1)} \beta_z^{(0)} \end{aligned} \quad (11)$$

$$\begin{aligned}
4.2. \quad & -\frac{\partial H_y^{(1)}}{\partial \tilde{z}} = \frac{\partial E_x^{(1)}}{\partial \tau} - \beta_0 \frac{\partial E_x^{(1)}}{\partial \tilde{z}} - n^{(0)} \beta_x^{(1)} \\
5. \quad & \frac{\partial E_x^{(1)}}{\partial \tilde{z}} - \frac{\partial E_z^{(1)}}{\partial x} = -\frac{\partial H_y^{(1)}}{\partial \tau} - \beta_0 \frac{\partial H_y^{(1)}}{\partial \tilde{z}}
\end{aligned}$$

System of eqs. (11) allowed to search the solution for $\rho_z^{(1)}, \rho_x^{(1)}, E_x^{(1)}, E_z^{(1)}, H_y^{(1)}$ in the forms ($\omega \equiv \frac{\omega_0}{\omega_p}, k \equiv \frac{k_0}{k_p}$):

$$\rho_z^{(1)} = \rho_{z0}^{(1)}(\tilde{z}) e^{-i\frac{\omega_0}{\omega_p}\tau + i\frac{k_0}{k_p}x} = \rho_{z0}^{(1)}(\tilde{z}) e^{-i\omega\tau + ikx} \quad (12)$$

(and subsequent expressions for other variables). Obtained set of equations will contain derivatives on \tilde{z} only, but still is quasilinear and complicated enough.

In order to simplicate this set of equations further, consider system (11), (12) behind the driving bunch in the region of wake wave spikes, where $\beta_z^{(0)}$ achived its maximum value $\beta_m > 0$:

$$\begin{aligned}
\beta_z^{(0)} &\approx \beta_m + \frac{1}{2} \left(\frac{\partial^2 \beta_z^{(0)}}{\partial \tilde{z}^2} \right)_m (\tilde{z} - \tilde{z}_m)^2, \\
|\tilde{z} - \tilde{z}_m|^2 &\ll 2\beta_m / \left(\frac{d^2 \beta_z^{(0)}}{d\tilde{z}^2} \right)_m
\end{aligned} \quad (13)$$

From eqs. (6), (7) for zero order approximation it follows that

$$\begin{aligned}
\left(\frac{d^2 \beta_z^{(0)}}{d\tilde{z}^2} \right)_m &= \frac{1}{(1 + \rho_m^2)^{3/2}} \left(\frac{d^2 \rho_z^{(0)}}{d\tilde{z}^2} \right)_m, \\
\left(\frac{d^2 \rho_z^{(0)}}{d\tilde{z}^2} \right)_m (\beta_0 - \beta_m) &= \left(\frac{\partial E_z^{(0)}}{\partial \tilde{z}} \right)_m = -(n_m - 1), \\
W \equiv \left(\frac{d^2 \beta_z^{(0)}}{d\tilde{z}^2} \right)_m &= -\frac{n_m(n_m - 1)}{\beta_0(1 + \rho_m^2)^{3/2}},
\end{aligned} \quad (14)$$

where ρ_m, n_m are maximun values of plasma electron momenta and density at wake wave spikes. If $\beta_m \rightarrow \beta_0, \rho_m \gg 1, n_m \gg 1$ and are near to wave breaking limit. The considered domain of the wake wave spikes (13), using (14), is approximately:

$$|\tilde{z} - \tilde{z}_m| \ll \left(\frac{2\beta_0^2 \rho_m^3}{n_m^2} \right)^{1/2} \approx \frac{2^{1/2} \gamma_0^{3/2}}{n_m} \quad (15)$$

In the domain, defined by (15), the system of eqs. (11), using (12), (13) can be essentially simplified and takes the following form:

$$\begin{aligned}
1.1. \quad & -i\omega\rho_{z0}^{(1)} + \beta_{z0}^{(1)}\frac{d\rho_z^{(0)}}{d\tilde{z}} = -E_{z0}^{(1)} - \mathcal{E}_{z0} \quad (16) \\
1.2. \quad & -i\omega\rho_{x0}^{(1)} = -E_{x0}^{(1)} + \beta_z^{(0)}(\mathcal{H}_{y0} + H_{y0}^{(1)}) \\
2. \quad & \frac{dE_{z0}^{(1)}}{d\tilde{z}} + ikE_{x0}^{(1)} = -n_0^{(1)} \\
3. \quad & -i\omega n_0^{(1)} + n^{(0)}\frac{d}{d\tilde{z}}\beta_{z0}^{(1)} + ikn^{(0)}\beta_{x0}^{(1)} = 0 \\
4.1. \quad & ikH_{y0}^{(1)} = -i\omega E_{z0}^{(1)} - \beta_0\frac{dE_{z0}^{(1)}}{d\tilde{z}} - n^{(0)}\beta_{z0}^{(1)} - n_0^{(1)}\beta_z^{(0)} \\
4.2. \quad & -\frac{dH_{y0}^{(1)}}{d\tilde{z}} = -i\omega E_{x0}^{(1)} - \beta_0\frac{dE_{x0}^{(1)}}{d\tilde{z}} - n^{(0)}\beta_{x0}^{(1)} \\
5. \quad & \frac{dE_{x0}^{(1)}}{d\tilde{z}} - ikE_{z0}^{(1)} = i\omega H_{y0}^{(1)} + \beta_0\frac{dH_{y0}^{(1)}}{d\tilde{z}}
\end{aligned}$$

The system of equations (16) is valid up to terms proportional to $(\tilde{z} - \tilde{z}_m)$, which are small according to assumption (15), and are disregarded in (16).

From eqs. (16.4.1) and (16.2) it follows

$$H_{y0}^{(1)} = -\frac{\omega}{k}E_{z0}^{(1)} + \beta_z^{(0)}E_{x0}^{(1)} + \frac{in^{(0)}}{k}\beta_{z0}^{(1)} \quad (17)$$

From eqs. (16.1.1), (16.1.2) and (17) it follows:

$$-i\omega\rho_{x0}^{(1)} = \beta_z^{(0)}\left(\mathcal{H}_{y0} + \frac{\omega}{k}\mathcal{E}_{z0}\right) + \frac{\omega}{k}\beta_z^{(0)}\left[-i\omega + \frac{in^{(0)}}{\omega}\frac{1}{(1 + \rho_z^{(0)2})^{1/2}}\right]\rho_{z0}^{(1)} \quad (18)$$

Substituting (17), (16.4.2) into (16.5) it is possible to obtain

$$E_{x0}^{(1)} = -\frac{k}{\omega}\left(1 - \frac{\omega^2}{k^2}\right)\frac{1}{(\beta_0 + \beta_z^{(0)})}E_{z0}^{(1)} - \frac{i\beta_{z0}^{(1)}n^{(0)}}{k(\beta_0 + \beta_z^{(0)})} + \frac{i\beta_0n^{(0)}\beta_{x0}^{(1)}}{\omega(\beta_0 + \beta_z^{(0)})} \quad (19)$$

In obtaining eqs. (17 - 19) the terms proportional to small quantities $\beta_0 - \beta_z^{(0)}$, $1 - \beta_0^2$, $1 - \beta_z^{(0)2}$ are omitted, due to relativism of driving bunch, conditions (13) and $\beta_z^{(0)} \rightarrow \beta_m \rightarrow \beta_0$.

From eqs. (16.2), (16.1.1), (18), (19), (10) it follows:

$$n_0^{(1)} = -\frac{n^{(0)}\rho_{z0}^{(1)}}{(\beta_0 + \beta_z^{(0)})(1 + \rho_z^{(0)2})^{3/2}} + \frac{k}{\omega}\frac{\beta_0n^{(0)}}{(\beta_0 + \beta_z^{(0)})(1 + \rho_z^{(0)2})^{1/2}} \times \quad (20)$$

$$\begin{aligned}
& \times \left[\frac{i\beta_z^{(0)}}{k} \left(\mathcal{E}_{z0} + \frac{k}{\omega} \mathcal{H}_{y0} \right) + \right. \\
& + \left. \left(\frac{\omega}{k} \beta_z^{(0)} - \frac{\beta_z^{(0)} n^{(0)}}{\omega k (1 + \rho_z^{(0)2})^{3/2}} \right) \rho_{z0}^{(1)} \right] - \frac{i}{\omega(\beta_0 + \beta_z^{(0)})} (k^2 - \omega^2) \mathcal{E}_{z0} - \\
& - \frac{(k^2 - \omega^2) \rho_{z0}^{(1)}}{(\beta_0 + \beta_z^{(0)})} - i\omega \frac{d\rho_{z0}^{(1)}}{d\tilde{z}} + \rho_{z0}^{(1)} \left(\frac{d^2 \beta_z^{(0)}}{d\tilde{z}^2} \right)_m
\end{aligned}$$

From continuity eq., written in the first approximation (16.3), using (10), (18) it is possible to obtain:

$$\begin{aligned}
n_0^{(1)} = & -\frac{in^{(0)}\beta_z^{(0)}}{\omega(1 + \rho_z^{(0)2})^{1/2}} \left[\left(\mathcal{E}_{z0} + \frac{k}{\omega} \mathcal{H}_{y0} \right) + \frac{in^{(0)}}{\omega} \frac{\rho_{z0}^{(1)}}{(1 + \rho_z^{(0)2})^{3/2}} - i\omega \rho_{z0}^{(1)} \right] - \\
& - \frac{in^{(0)}}{\omega(1 + \rho_z^{(0)2})^{3/2}} \frac{d\rho_{z0}^{(1)}}{d\tilde{z}}
\end{aligned} \quad (21)$$

From eqs. (20) and (21) it follows equation for $\rho_{z0}^{(1)}$, where it is possible to take $\beta_z^{(0)} \approx \beta_m \approx \beta_0$

$$\begin{aligned}
& -i\omega \left(1 - \frac{n^{(0)}}{\omega^2 (1 + \rho_z^{(0)2})^{3/2}} \right) \frac{d\rho_{z0}^{(1)}}{d\tilde{z}} + \\
& + \left[\frac{n^{(0)2}\beta_0}{2\omega^2 (1 + \rho_z^{(0)2})^2} - \frac{n^{(0)}}{2\beta_0 (1 + \rho_z^{(0)2})^{3/2}} - \right. \\
& - \frac{n^{(0)}\beta_0}{2(1 + \rho_z^{(0)2})^{1/2}} + \left. \left(\frac{d^2 \beta_z^{(0)}}{d\tilde{z}^2} \right)_m - \frac{(k^2 - \omega^2)}{2\beta_0} \right] \rho_{z0}^{(1)} = \\
& = \frac{in^{(0)}\beta_0}{2\omega(1 + \rho_z^{(0)2})^{1/2}} \left(\mathcal{E}_{z0} + \frac{k}{\omega} \mathcal{H}_{y0} \right) + \frac{i(k^2 - \omega^2)}{2\omega\beta_0} \mathcal{E}_{z0}
\end{aligned} \quad (22)$$

where $\mathcal{E}_{z0}, \mathcal{H}_{y0}$ are amplitudes of external EM-wave, given by (1).

4 Estimates of Amplification Factors

Coefficients of eq. (22) can be estimated using conditions (13, 14, 15) and in the spikes region defined by (15) $n^{(0)} \rightarrow n_m \gg 1, \rho_z^{(0)} \rightarrow \rho_m \gg 1$ and for

$\beta_0 \rightarrow 1$ eq. (22) takes simple form

$$A \frac{d\rho_{z0}^{(1)}}{d\tilde{z}} + P\rho_{z0}^{(1)} = Q, \quad (23)$$

where

$$\begin{aligned} A &\equiv -i\omega \left(1 - \frac{n_m}{\omega^2 \rho_m^3}\right) \\ P &\equiv \frac{n_m^2}{2\omega^2 \rho_m^4} - \frac{n_m}{2\rho_m^3} - \frac{n_m}{2\rho_m} - \frac{n_m(n_m - 1)}{\rho_m^3} - \frac{1}{2}(k^2 - \omega^2) \\ Q &\equiv \frac{in_m}{\omega \rho_m} \left(\mathcal{E}_{z0} + \frac{k}{\omega} \mathcal{H}_{y0}\right) + \frac{i}{2\omega}(k^2 - \omega^2)\mathcal{E}_{z0} \end{aligned} \quad (24)$$

Asymptotic particular solution of eq. (23), which in the spikes region (15) is independent on \tilde{z} is given by

$$\rho_{z0}^{(1)} \approx Q/P \quad (25)$$

The last term in P in (24) is

$$-(k^2 - \omega^2) = \left(\frac{\omega_0}{\omega_p}\right)^2 (1 - \epsilon) = \frac{\varpi_p^2}{\omega_p^2} \left(1 - \epsilon'' \frac{\omega_0^2}{\varpi_p^2}\right) \approx \frac{\varpi_p^2}{\omega_p} = n_m \gg 1 \quad (26)$$

and it is much larger than all other terms in P, and when $n_m \gg, \rho_m \gg 1, \beta_0 - \beta_m \gg \gamma_0^{-3}$

$$P \approx \frac{1}{2}n_m, Q \approx i \left(\frac{n_m \varpi_p^2}{\rho_m \omega_0} - \frac{\omega_p n_m}{\omega_0}\right) \mathcal{E}_{z0} \quad (27)$$

From (25) then

$$\rho_{z0}^{(1)} = 2i\mathcal{E}_{z0} \frac{\omega_p}{\omega_0} \left(\frac{n_m \omega_p}{\rho_m \omega_0} - 1/2\right) \quad (28)$$

Consider two limiting cases of obtained solution (28).

Case (a):

$$\frac{\rho_m \omega_0}{n_m \omega_p} < \frac{\rho_0 \omega_0}{n_m \omega_p} = \frac{(\beta_0 - \beta_m) \omega_0}{(1 - \beta_0^2)^{1/2} \omega_p} \ll 1, \quad (29)$$

i.e.

$$\beta_0 - \beta_m \ll \frac{\omega_p}{\omega_0} \gamma_0^{-1}, n_m = \frac{\beta_0}{\beta_0 - \beta_m} \gg \frac{\omega_0}{\omega_p} \gamma_0 \quad (30)$$

Simultaneous fulfillment of the condition of plasma spikes transparency $\frac{\omega_p^2}{\omega_0^2} < 1, n_m < \frac{\omega_0^2}{\omega_p^2}$ could take place if:

$$1 \ll \frac{\omega_0}{\omega_p} \gamma \ll n_m < \frac{\omega_0^2}{\omega_p^2}, 1 \ll \gamma_0 \ll \frac{\omega_0}{\omega_p}, \quad (31)$$

$$\frac{\omega_p^2}{\omega_0^2} < \beta_0 - \beta_m \ll \frac{\omega_p}{\omega_0} \gamma_0^{-1};$$

Case (b):

$$\frac{n_m \omega_p}{\rho_m \omega_0} \ll 1 \quad (32)$$

It is opposite to the case (a) condition and it takes place, when

$$\beta_0 - \beta_m \gg \frac{\omega_p}{\omega_0} \gamma_0^{-1}, \beta_0 - \beta_m > \frac{\omega_p^2}{\omega_0^2} \quad (33)$$

$$1 \ll n_m \ll \frac{\omega_0}{\omega_p} \gamma_0, 1 \ll n_m \leq \frac{\omega_0^2}{\omega_p^2}$$

In the case (b) no additional restriction on γ_0 is needed, the relativistic condition $\gamma_0 \gg 1$ remains valid.

Wake spikes domain, given by (15), when n_m is taken by order of magnitude equal ω_0^2/ω_p^2 is

$$|\tilde{z} - \tilde{z}_m| \ll 2^{1/2} \gamma_0^{3/2} \frac{\omega_p^2}{\omega_0^2} \quad (34)$$

and in ordinary units is much smaller than $\lambda_p/2\pi$ in the case (a) and in the case (b) could be by order of magnitude equal to $\lambda_p/2\pi$, where λ_p is plasma linear wave length.

Electric field z - component inside the plasma wake wave spikes (15), using (16.1.1), (28) is

$$E_{z0}^{(1)} \approx -\mathcal{E}_{z0} - \frac{i\omega_0}{\omega_p} \rho_{z0}^{(1)} = -2\mathcal{E}_{z0} \left(1 - \frac{n_m \omega_p}{\rho_m \omega_0} \right) \quad (35)$$

and for the case (a) is given by

$$E_{z0}^{(1)} \approx 2\mathcal{E}_{z0} \left(\frac{n_m \omega_p}{\rho_m \omega_0} \right) \gg \mathcal{E}_{z0} \quad (36)$$

For the case (b):

$$E_{z0}^{(1)} \approx -2\mathcal{E}_{z0} \quad (37)$$

and field amplification factors are

$$K_z = \frac{|E_{z0}^{(1)}|}{|\mathcal{E}_{z0}|}, K_z^a = 2 \left(\frac{n_m \omega_p}{\rho_m \omega_0} \right) \gg 1, K_z^b = 2 \quad (38)$$

for the case (a) and case (b) correspondingly.

In order to obtain x - component of the electric field inside the spikes it is necessary to use eqs. (18), (19). Assuming that ratio $\frac{n_m \omega_p}{\rho_m^3 \omega_0}$ is small for $\rho_m \gg 1$ and using (28) it is possible to obtain

$$\begin{aligned} \rho_{x0}^{(1)} &\approx \frac{1\rho_0}{k} \left(\mathcal{E}_{z0} + \frac{k}{\omega} \mathcal{H}_{z0} \right) + \beta_0 \omega / k \rho_{z0}^{(1)} \approx \\ &\approx \frac{i}{\sqrt{\epsilon}} \frac{\omega_p}{\omega_0} (1 - \epsilon') \mathcal{E}_{z0} + \frac{1}{\sqrt{\epsilon}} \rho_{z0}^{(1)} = \\ &= \frac{i}{\sqrt{\epsilon}} \mathcal{E}_{z0} \left\{ n_m \left(\frac{\omega_p}{\omega_0} \right)^3 + \frac{2\omega_p}{\omega_0} \left(\frac{n_m \omega_p}{\rho_m \omega_0} - 1/2 \right) \right\} \end{aligned} \quad (39)$$

From eq (19) at the same condition it follows

$$E_{x0}^{(1)} = -\frac{1}{2\beta_0 k \omega} (k^2 - \omega^2) E_{z0}^{(1)} + \frac{i}{\omega} \frac{n^{(0)} \rho_{x0}^{(1)}}{2(1 + \rho_z^{(0)2})^{1/2}} \quad (40)$$

and using (35), (39) after some transformation it is possible to obtain

$$E_{x0}^{(1)} = -\frac{\mathcal{E}_{z0}}{2\sqrt{\epsilon}} \frac{\omega_p}{\omega_0} \left[2n_m \left(\frac{\omega_p}{\omega_0} \right) \left(1 - \frac{n_m \omega_p}{\rho_m \omega_0} \right) - \frac{n_m \omega_p}{\rho_m \omega_p} \left(n_m \frac{\omega_p^2}{\omega_0^2} + 2 \frac{n_m \omega_p}{\rho_m \omega_0} - 1 \right) \right] \quad (41)$$

For the case (a), when $\frac{n_m \omega_p}{\rho_m \omega_0} \gg 1$, $\frac{\omega_p}{\omega_0} \gg 1$ and $n_m \leq \frac{\omega_0}{\omega_p}$

$$E_{x0}^{(1)} \approx \frac{\mathcal{E}_{z0}}{\sqrt{\epsilon}} \frac{n_m^2}{\rho_m} \left(\frac{\omega_p}{\omega_0} \right)^3 = \frac{\mathcal{E}_{z0}}{\sqrt{\epsilon}} \left(\frac{n_m \omega_p}{\rho_m \omega_0} \right) n_m \left(\frac{\omega_p}{\omega_0} \right)^2 \leq \frac{\mathcal{E}_{z0}}{\sqrt{\epsilon}} \left(\frac{n_m \omega_p}{\rho_m \omega_0} \right) \quad (42)$$

and amplification factor

$$K_x^a = \frac{|E_{x0}^{(1)}|}{|\mathcal{E}_{z0}|} = \frac{1}{\sqrt{\epsilon}} \frac{n_m^2}{\rho_m} \left(\frac{\omega_p}{\omega_0} \right)^3 \quad (43)$$

could be large enough, if

$$\left(\frac{\omega_p}{\omega_0} \right)^2 < (\beta_0 - \beta_m) \ll \min \left\{ \frac{1}{\gamma^{1/2}} \left(\frac{\omega_p}{\omega_0} \right)^{3/2}, \frac{1}{\gamma} \left(\frac{\omega_p}{\omega_0} \right) \right\} \quad (44)$$

Conditions (44) included conditions (30) for the realization of the case (a). For the case (b), when $\left(\frac{n_m \omega_p}{\rho_m \omega_0} \right) \ll 1$ and $1 \ll n_m \frac{\omega_p}{\omega_0} \ll \rho_m$:

$$E_{x0}^{(1)} = -\frac{\mathcal{E}_{z0}}{\sqrt{\epsilon}} n_m \left(\frac{\omega_p}{\omega_0} \right)^2 \quad (45)$$

and corresponding field amplification factor is

$$K_x^b = \frac{1}{\sqrt{\epsilon}} n_m \left(\frac{\omega_p}{\omega_0} \right)^2, n_m \leq \left(\frac{\omega_0}{\omega_p} \right)^2 \quad (46)$$

In all considered cases it is necessary to be sure that adopted perturbative approach is not violated. In particular, the conditions for applicability of the decompositions (9), (10) must be fulfilled

$$|\rho_{z0}^{(1)}| \ll |\rho_z^{(0)}|, |\rho_{x0}^{(1)}| \ll |\rho_z^{(0)}| \quad (47)$$

In the considered plasma wake wave spikes domain (15) the conditions (47) using (4), (28) can be rewritten as

$$\rho_m \gg \left| 2a \left(\frac{n_m \omega_p}{\rho_m \omega_0} - 1/2 \right) \right| \quad (48)$$

For the case (a) it means that:

$$1 \ll \frac{n_m \omega_p}{\rho_m \omega_0} \ll \frac{\rho_m}{2a}, a \equiv \frac{e \mathcal{E}_0}{mc \omega_0} \ll 1 \quad (49)$$

$$a \ll \frac{\rho_m^2 \omega_0}{2n_m \omega_p}$$

For the case (b) condition (48) just gives

$$a \ll \rho_m, \rho_m \gg 1 \quad (50)$$

Turning now to estimates of amplification factors for EM-wave, Thomson scattered on plasma spikes, consider the plasma current densities on spikes, which in the first approximation of considered perturbative approach are given by

$$j_{x0}^{(1)} = n^{(0)} \beta_{x0}^{(1)}, j_{z0}^{(1)} = n^{(0)} \beta_{z0}^{(1)} + n_0^{(1)} \beta_z^{(0)} \quad (51)$$

In considered plasma wake wave spikes domain (15):

$$n^{(0)} \approx n_m, \beta_z^{(0)} \approx \beta_m \approx \beta_0 \approx 1, \beta_{z0}^{(1)} \approx \frac{\rho_{z0}^{(1)}}{\rho_m^3}, \beta_{x0}^{(1)} \approx \frac{\rho_{x0}^{(1)}}{\rho_m} \quad (52)$$

From eq. (21) for $n_0^{(1)}$ it follows that the main contribution to $n_0^{(1)}$ came in the case (a) from the last term in the square bracket in (21) and, using (28) for the case (a), obtain:

$$n_0^{(1)} \approx -\frac{n^{(0)} \beta_z^{(0)}}{(1 + \rho_z^{(0)2})^{1/2}} \rho_{z0}^{(1)} \approx -2i \left(\frac{n_m \omega_p}{\rho_m \omega_0} \right)^2 \mathcal{E}_{z0} \quad (53)$$

For considered current densities (51) using (52, 53, 28, 39) the following estimates for the case (a) can be obtained:

$$\begin{aligned} j_{x0}^{(1)} &\approx n_m \frac{\rho_{x0}}{\rho_m} \approx \frac{2}{\sqrt{\epsilon}} \left(\frac{n_m \omega_p}{\rho_m \omega_0} \right)^2 \mathcal{E}_{z0} \\ j_{z0}^{(1)} &\approx n_0^{(1)} \approx -2i \left(\frac{n_m \omega_p}{\rho_m \omega_0} \right)^2 \mathcal{E}_{z0} \end{aligned} \quad (54)$$

The radiated outside the plasma Thomson scattered EM-wave is described by potentials:

$$A_{x,y} = \frac{1}{cR_0} \int j_{x,y}(x, y, z, t - \frac{\vec{r}\vec{n}}{c}) dv \sim \frac{j_{0x,y}^{(1)}}{cR_0}, \quad (55)$$

where R_0 is the distance from radiator (spikes) to observation point outside the plasma, $r^2 = x^2 + y^2 + z^2$, \vec{n} - unit vector in direction of radiation.

Radiated energy flux in the unit solid angle per second from unit volume of plasma spike is proportional to

$$\frac{dW_{x,z}}{d\Omega} \sim |j_{0x,z}^{(1)}|^2$$

and corresponding intensity amplification factor

$$K_{radx,z} = \frac{1}{W_0} \frac{dW_{x,z}}{d\Omega} \sim \frac{|j_{0x,z}|^2}{|\mathcal{E}_{z0}|^2} \quad (56)$$

In (56) W_0 is the incident energy flux of external EM - wave (1) on unit area of plasma wake wave spike cross section normal to 0x -axis (direction of EM - wave propagation) per second, $W_0 \sim |\mathcal{E}_{z0}|^2$. From (56), (54) it follows that

$$K_{radx}^a \sim \frac{4}{|\epsilon|} \left(\frac{n_m \omega_p}{\rho_m \omega_0} \right)^4 \gg 1 \quad (57)$$

$$K_{radz}^a \sim 4 \left(\frac{n_m \omega_p}{\rho_m \omega_0} \right)^4 \gg 1 \quad (58)$$

The field amplification factors (43), (46) and intensity amplification factor (58), attributed to the plasma electron motion in plasma wave spikes along x - axis posses multipliers $|\epsilon|^{-1/2}$ and $|\epsilon|^{-1}$ subsequently. It means that at resonance conditions, when frequency of external EM - wave is nearing the plasma frequency at the nonlinear wave spikes $\omega_0^2 \rightarrow \varpi_p^2 = n_m \omega_p^2$ an additional amplification can take place. Additional amplification factors are equal, according to (2):

$$|\epsilon|^{-1/2} = \frac{1}{\sqrt{|\epsilon''|}} = \left(\frac{\omega_0}{\nu_{eff}} \right)^{1/2} \quad (59)$$

$$|\epsilon|^{-1} = \frac{1}{\sqrt{|\epsilon''|}} = \left(\frac{\omega_0}{\nu_{eff}} \right), \omega_0 \rightarrow \varpi_p, \nu_{eff} \ll \omega_p \ll \omega_0$$

Mentioned resonance amplification (59) takes place, of course independantly from conditions (30), (33) of realization the cases (a) or (b).

5 Conclusion

The obtained analytical estimates demonstrate, that at certain conditions on cold plasma - relativistic electron bunch - external EM - wave system parameters (see (30), (33), (59)), essentially large amplification of electric field inside the plasma spikes, as well as intensity of Thomson scattered on spikes EM - wave, are existed (see expression for amplification factors (38), (43), (46), (57), (58), (59)). Presented results could be used in research and development of powerful klystron type amplifiers and generators of high frequency EM - waves for future linear colliders.

The presented in the work estimates have at some extent qualitative character and must be complemented by more quantitative investigations, presumably by computer simulations. The reason for computer calculations is evident from the fact, that even in perturbative approach, addopted in present work, the problem is reduced, due to nonlinearity of zero order approximation, to the set of quasilinear equations (11), with variable coefficients, nonlinearly depending on argument \tilde{z} . The exact analytical solution of the set of equations (11) is practically impossible to obtain.

However, more elaborate perturbative approach, for example based on multiple scales method, could provide the possibility to go further in directions outlined in the present work.

In order to find out the optimal geometry for experimental detection of the predicted EM - wave amplification, it is necessary to consider also different directions of propagation and different polarizations of external EM - wave.

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